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The basis of three-particle hyperspherical harmonics in 'democratic' variables

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Abstract. A complete system of three-particle hyperspherical harmonics (HH) with orbital momenta $L \leqslant 4$ is constructed in an explicit form with separated rotational degrees of freedom. Formulae for any L values are given. Permutational and other properties of HH and a method of HH orthonormalisation are considered.

1. Introduction

Let Ω be a hypersphere $\rho=(x^2+y^2)^{1/2}=\rho_0$ in a six-dimensional space $R_6=\{x, y\}$. Hyperspherical harmonics (HH) $Y_K(\Omega)$ are defined by the following two conditions: $\rho^K Y_K(\Omega)$ is a degree K polynomial and if $K \neq K'$

$$\int \mathrm{d}\Omega\ Y_K^*(\Omega) Y_{K'}(\Omega)=0$$

($\mathrm{d}\Omega=\rho_0^{-5}\delta(\rho-\rho_0)\,\mathrm{d}x\,\mathrm{d}y$). As is well known the $Y_K(\Omega)$ functions are useful in many cases for the investigation and solution of dynamical three-body problems in quantum mechanics (cf e.g. Fano 1983) and for phenomenological analysis of three-particle reaction amplitudes. In such problems the above-mentioned vectors x, y mean the three-particle Jacobi coordinates

$$_k\xi_1=\left(\frac{m_i m_j}{(m_i+m_j)\mu_0}\right)^{1/2}(r_j-r_i) \qquad {}_k\xi_2=\left(\frac{(m_i+m_j)m_k}{(m_i+m_j+m_k)\mu_0}\right)^{1/2}\left(r_k-\frac{m_i r_i+m_j r_j}{m_i+m_j}\right)$$

$$({}_k\xi_1^2+{}_k\xi_2^2={}_i\xi_1^2+{}_i\xi_2^2={}_j\xi_1^2+{}_j\xi_2^2=\rho^2)$$

or the conjugated Jacobi momenta. In a relativistic three-particle amplitude case angular variables defined on Ω parametrise a hypersurface that corresponds to some fixed energy-momentum values and corresponding x, y vectors are only auxiliary (cf Efros 1983).

It is convenient to specify HH along with the K number by the rotational quantum numbers L and M. Five angular variables which parametrise the Ω hypersphere may include three Euler angles $\{\omega\}$ specifying space orientation of a three-particle system as a whole. For many applications it is convenient to construct HH as an expansion in corresponding Wigner $\mathscr{D}$ functions $\mathscr{D}^L_{MM'}(\omega)$. A list of references concerning the construction of such HH may be found in Mukhtarova and Efros (1983). In particular, large tables of such HH with $L \leqslant 4$ values for various fixed K values have been reported (Del Aguila and Doncel 1980). However, general analytic formulae for HH with arbitrary

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K values would be more convenient. Since L is generally related to integrals of motion of a problem and takes a limited set of values, formulae with fixed L values are convenient. For the $L=0$ case such formulae were obtained by Gronwall (1937), for the $L=1$ case by Zickendraht (1965) and Levy-Leblond and Levy-Nahas (1965). For the $L=2$ case such formulae were obtained by Zickendraht (1965), however, within the framework of non-systematic HH classification. Using the approach described by Mukhtarova and Efros (1983), such formulae are obtained below for all $L \leqslant 4$ values, which is sufficient for the majority of applications. We use a systematic HH classification described in § 2. In § 3 the main properties of the HH are described, with explicit expressions for the HH basis given in § 4 and a conclusion in § 5.

2. General formulae

The five angular variables parametrising the Ω hypersphere include Euler angles $\{\omega\}$ and two scalar variables λ and ψ. The $\{\omega\}$ angles are defined in a standard way by an orientation of unit vectors attached to a three-particle system relative to a laboratory coordinate system. Let $z=y+ix$, where x, y are the vectors mentioned above. Then the λ, ψ variables and the unit vectors e_i are defined by the relations

$$\begin{aligned} &\sin\psi \exp(i\lambda)=\rho^{-2}z^2 \\ &e_1=\tfrac{1}{2}[\rho\cos(\tfrac{1}{2}\psi-\tfrac{1}{4}\pi)]^{-1}[z\exp(-i\lambda/2)+z^*\exp(i\lambda/2)] \\ &e_2=\tfrac{1}{2}[\rho\cos(\tfrac{1}{2}\psi+\tfrac{1}{4}\pi)]^{-1}i[z\exp(-i\lambda/2)-z^*\exp(i\lambda/2)] \\ &e_3=[e_1,e_2]=(\rho^2\cos\psi)^{-1}2[x,y] \end{aligned} \tag{1}$$

with $0\leqslant\psi\leqslant\frac{1}{2}\pi$, $0\leqslant\frac{1}{2}\lambda\leqslant\pi$. We have $d\Omega=\frac{1}{8}\sin\psi\cos\psi\, d\psi\, d\lambda\, d\omega$. The $\{\omega\}$, λ, ψ variables (the so-called 'democratic' variables (Smith 1960, 1962)) are simply transformed under particle permutations (see e.g. Mukhtarova and Efros 1983).

A general form of the HH required is as follows (Zickendraht 1965, Levy-Leblond and Levy-Nahas 1965):

$$Y^{\alpha\nu}_{KLM}(\Omega)=e^{i\lambda\nu}\sum_{M'=-L}^{L}\mathscr{D}^{L*}_{MM'}(\omega)f^{\alpha\nu}_{KLM'}(\psi) \tag{2}$$

where M' takes only values of the same parity as the K value. The $f(\psi)$ functions are to be determined. When applying an orthogonal transformation

$$x\to x\cos\varphi+y\sin\varphi \qquad y\to -x\sin\varphi+y\cos\varphi \tag{3}$$

we have, as seen from (1), $\psi\to\psi$, $e_i\to e_i$, $\lambda\to\lambda+2\varphi$. This transformation leads to $Y^{\alpha\nu}_{KLM}\to Y^{\alpha\nu}_{KLM}\exp(2i\nu\varphi)$ in (2) that elucidates the meaning of the index ν.

The HH of the equation (2) form which we are constructing are defined by the condition that up to a multiplier, $\rho^K Y^{\alpha\nu}_{KLM}$ polynomials with $M=L$ are harmonic projections (Vilenkin 1965) of the polynomials

$$\mathscr{Y}^{pq}_{LM}(z,z^*)(z^2)^r(z^{*2})^s \tag{4}$$

with

$$\mathscr{Y}^{pq}_{LM}(z,z^*)=\sum_{m_1+m_2=M}(pm_1qm_2|LM)\mathscr{Y}_{pm_1}(z)(\mathscr{Y})_{qm_2}(z^*) \tag{5}$$

$$K=(p+q)+(2r+2s) \qquad 2\nu=(p-q)+(2r-2s) \qquad 2\alpha=p-q \tag{6}$$

$$L = \begin{cases} p+q & K-L \text{ even} \quad (7) \\ p+q-1 & K-L \text{ odd.} \quad (8) \end{cases}$$

We have $K \geqslant L$ or $K \geqslant L+1$ in the case of equations (7) and (8) respectively. For $L=0$, K values may be even only. Equation (6) elucidates the meaning of the index α in (2). As seen from (6), the α values that are possible for a given L value are as follows: $2\alpha = L, L-2, \ldots, -(L-2), -L$ in the case of equation (7) and $2\alpha = L-1, L-3, \ldots, -(L-3), -(L-1)$ in the case of equation (8). The ν values possible for given K, L, α values may be readily obtained from (6) if one writes $2\nu = 2\alpha + 2\nu_s$ taking into account that the $2\nu_s$ values are equal to $K-L, K-L-4, \ldots, -(K-L-4), -(K-L)$ in the case of equation (7) and that they are equal to $(K-L-1), (K-L-5), \ldots, -(K-L-5), -(K-L-1)$ in the case of equation (8).

3. Symmetrisation, orthogonalisation and reality properties

One can see from (4)-(6) that

$$Y^{\alpha\nu}_{KLM|z\rightleftarrows z^*} = (-1)^{K-L} Y^{-\alpha,-\nu}_{KLM}. \tag{9}$$

Performing the transformation $z \rightleftarrows z^*$ in (1) and (2) and comparing the result with (9) we obtain

$$f^{-\alpha,-\nu}_{KL,-M'}(\psi) = (-1)^K f^{\alpha\nu}_{KLM'}(\psi). \tag{10}$$

On the other hand, from (4) and (6) it follows that

$$[Y^{\alpha\nu}_{KLM}(\Omega)]^* = (-1)^M Y^{-\alpha,-\nu}_{KL,-M}(\Omega). \tag{11}$$

The comparison of (11) with (10) shows the functions $f^{\alpha\nu}_{KLM'}$ to be real.

In the case of two identical particles it is convenient to use linear combinations of the HH of (2) with the required symmetry properties

$$Y^{(+)\alpha\nu}_{KLM} = \tfrac{1}{2}[1+(\widehat{ij})] Y^{\alpha\nu}_{KLM} \qquad Y^{(-)\alpha\nu}_{KLM} = \mathrm{i}^{-1}\tfrac{1}{2}[1-(\widehat{ij})] Y^{\alpha\nu}_{KLM}. \tag{12}$$

Here $(\widehat{ij})$ is a particle transposition. Taking into account the relations $(\widehat{ij})z = z^*$, $(\widehat{ij})z^* = z$ and (9), one may write (12) as follows:

$$\begin{aligned} Y^{(+)\alpha\nu}_{KLM} &= \tfrac{1}{2}[Y^{\alpha\nu}_{KLM} + (-1)^{K-L} Y^{-\alpha,-\nu}_{KLM}] \\ Y^{(-)\alpha\nu}_{KLM} &= \mathrm{i}^{-1}\tfrac{1}{2}[Y^{\alpha\nu}_{KLM} - (-1)^{K-L} Y^{-\alpha,-\nu}_{KLM}]. \end{aligned} \tag{13}$$

The right-hand side of (13) includes the α, ν and $-\alpha, -\nu$ indices simultaneously. Therefore, to obtain a complete system of the HH it is sufficient to confine ourselves for example to the HH with $\nu \geqslant 0$ in the right-hand side of (12) with the additional condition $\alpha \geqslant 0$ for the $\nu = 0$ case. It will be used below.

In the case of all three particles being identical the HH of (12) and (13) are known to form irreducible representations of a three-particle permutation group. In the $2\nu = 0 \pmod 3$ cases $Y^{(+)\alpha\nu}_{KLM}$ is a symmetrical function and $Y^{(-)\alpha\nu}_{KLM}$ is an antisymmetrical function with respect to three-particle permutations. In the $2\nu = 1 \pmod 3$ and $2\nu = 2 \pmod 3$ cases the pair of functions $(Y^{(+)\alpha\nu}_{KLM}, Y^{(-)\alpha\nu}_{KLM})$ form the two-dimensional representation of a permutation group. The standard basis of this representation (Hamermesh 1964) corresponds to the $(Y^{(+)\alpha\nu}_{KLM}, Y^{(-)\alpha\nu}_{KLM})$ pair in the $2\nu = 1 \pmod 3$ case and to the $(Y^{(+)\alpha\nu}_{KLM}, -Y^{(-)\alpha\nu}_{KLM})$ pair in the $2\nu = 2 \pmod 3$ case. This can easily be seen if one takes into account that the cyclic permutation (123) is a particular case of the transformation of (3).

Taking (11) into account, it is easy to find that for the basis of (13), the matrix elements (ME) of the parity and rotation invariant Hamiltonian are real†. The same also holds true if spin variables are involved provided the ME between even and odd L values vanish. And if they do not vanish all the ME may be made real, for example, by making the redefinition $Y^{\alpha\nu}_{KLM} \to \mathrm{i}\, Y^{\alpha\nu}_{KLM}$ for odd L values and using the functions of type (13). The transition from the HH of (2) to that of (13) ensures the ME mentioned to be real in the non-identical particle case as well.

The HH of (2) are mutually orthogonal if some of their indices K, L, M, ν are different. The same holds true for the HH of (13) if $\nu \geqslant 0$ is meant in the designation of $Y^{(\pm)\alpha\nu}_{KLM}$ as was said above. (The number of HH in each group of mutually non-orthogonal HH with the same K, L, M, ν values is approximately equal to L.)

To perform a complete orthonormalisation of the HH set $Y^{(\pm)\alpha\nu}_{KLM}$ it is sufficient to calculate their overlap integrals (OI) with different α values. These OI are expressed with the OI

$$\int Y^{\alpha'\nu *}_{KLM}(\Omega)\, Y^{\alpha\nu}_{KLM}(\Omega)\, \mathrm{d}\Omega = [\alpha'\alpha;\, K\nu L] \tag{14}$$

for the HH of (2). We have

$$\begin{aligned} &\int Y^{(+)\alpha'\nu *}_{KLM}(\Omega)\, Y^{(-)\alpha\nu}_{KLM}(\Omega)\, \mathrm{d}\Omega = 0 \\ &\int Y^{(\pm)\alpha'\nu *}_{KLM}(\Omega)\, Y^{(\pm)\alpha\nu}_{KLM}(\Omega)\, \mathrm{d}\Omega = \begin{cases} [\alpha'\alpha;\, K\nu L] & \nu > 0 \\ \tfrac{1}{2}([\alpha'\alpha;\, K0L] \pm [\alpha', -\alpha;\, K0L]) & \nu = 0. \end{cases} \end{aligned} \tag{15}$$

The orthonormalised HH may be written, for example, as determinants involving the HH $Y^{(\pm)\alpha\nu}_{KLM}$ and the OI of (15) (*Higher transcendental functions* 1953, § 10.1).

To calculate the OI of (14) we first integrate in $\mathrm{d}\lambda$ and $\mathrm{d}\omega$

$$\begin{aligned} &[\alpha'\alpha;\, K\nu L] = \int_{-1}^{1} \mathrm{d}t\, F_n(t) \\ &F_n(t) = \tfrac{1}{2}\pi^3 \sum_{M'=-L}^{L} f^{\alpha'\nu}_{KLM'}(\psi) f^{\alpha\nu}_{KLM'}(\psi) \end{aligned} \tag{16}$$

where $t = \cos(2\psi)$. It is not difficult to show that $F_n(t)$ is a polynomial in t of $n = [K/2]$ degree ([...] denotes an integer part). Since the integrand is a polynomial the exact value of OI in (16) is given by the relation

$$[\alpha'\alpha;\, K\nu L] = \sum_{i=1}^{N} w_i F_n(t_i)$$

where t_i and w_i are abscissas and weights of the Gauss quadrature formula and the N value is an integer exceeding $\frac{1}{2}(n-1)$. On the other hand, for values of $L \leqslant 3$, OI of (15) were calculated by Mukhtarova and Efros (1979) in an explicit form‡. The HH of (2) differ from that of Mukhtarova and Efros (1979) by a constant factor and

†The ME independence on M should be used and the substitution $M \to -M$ should be made.

‡There is a misprint in OI (11; $K\nu 2$) of that paper. The $(K+2\nu)$ factor should be placed outside the braces.

to use OI from that paper we list the relations between them, denoted there as $(\alpha'\alpha;\ K\nu L)$, and the OI of (14)

$$[\alpha'\alpha;\ K\nu L]=C(K,L)(\alpha'\alpha;\ K\nu L) \qquad K-L \text{ even}$$

$$[\alpha'\alpha;\ K\nu L]=\gamma(\alpha)\gamma(\alpha')C(K,L-1)(\alpha'\alpha;\ K\nu L) \qquad K-L \text{ odd}$$

$$C(K,L)=2\pi^3(K-L+2)^{-1}\{(K+1)![2^L(K-L+1)!]^{-1}\}^2$$

$$\gamma(\alpha)=\tfrac{1}{4}[(K-L+3)^2-(2\nu-2\alpha)^2]^{1/2}.$$

4. Explicit expressions for the HH basis

Explicit expressions for the required $f^{\alpha\nu}_{KLM'}(\psi)$ functions from (2) for $L\leqslant 4$ values are given in table 1 for the even $K-L$ case and in table 2 for the odd $K-L$ case. The formulae are listed for values of $\alpha\geqslant 0$. For values of $\alpha<0$ equation (10) may be used. The known results for the $L=0$ and $L=1$ cases are also listed for the sake of completeness. The $f^{\alpha\nu}_{KLM'}$ functions in the $L=0$ and $L=1$ and even K cases are written explicitly. In other cases some additional notation is introduced. For a given L value the $f^{\alpha\nu}_{KLM'}$ functions are written in the tables in terms of $G^{\alpha}_{\varkappa}(\psi)$ and $s_i(\psi)$ functions. The latter functions are defined as $s_i=[\cos(\psi/2)]^{L+1-i}[\sin(\psi/2)]^{i-1}$, $i=1,\ldots,L+1$ in the case of table 1 and $s_i=[\cos(\psi/2)]^{L-i}[\sin(\psi/2)]^{i-1}$, $i=1,\ldots,L$ in the case of table 2. The functions $G^{\alpha}_{\varkappa}(\psi)$ are defined as

$$G^{\alpha}_{\varkappa}(\psi)=(-1)^n(\sin\psi)^{|\nu-\varkappa|}g^{\alpha}_{\varkappa}(\psi) \tag{17}$$

whereas the $g^{\alpha}_{\varkappa}$ functions are listed in the tables explicitly. The $\varkappa$ index is equal to $\varkappa=-L/2,\ -L/2+1,\ldots,L/2$ for even $K-L$ and to $\varkappa=-(L-1)/2,\ -(L-1)/2+1,\ldots,(L-1)/2$ for odd $K-L$. The α indices for $g^{\alpha}_{\varkappa}$ functions are listed in the left-hand columns of the tables. In the formulae defining the $g^{\alpha}_{\varkappa}$ functions in the tables the following notations are used: $n=n(\varkappa)$ (it occurs also in (17)), $l=l(\varkappa)$, l_1, l_2, l_3 and P^{β}_{N}. They are defined as follows: for even $K-L$ values

$$n=\tfrac{1}{4}(K+L)-\tfrac{1}{2}(|\nu-\varkappa|+\varepsilon) \qquad l=\tfrac{1}{4}(K+L)+\tfrac{1}{2}(|\nu-\varkappa|+\varepsilon). \tag{18a}$$

For odd $K-L$ values

$$n=\tfrac{1}{4}(K+L+1)-\tfrac{1}{2}(|\nu-\varkappa|+\varepsilon) \qquad l=\tfrac{1}{4}(K+L+1)+\tfrac{1}{2}(|\nu-\varkappa|+\varepsilon). \tag{18b}$$

In (18a) and (18b) $\varepsilon=0$ if $\alpha+\varkappa$ is even and $\varepsilon=1$ if $\alpha+\varkappa$ is odd. Also

$$l_1=l(l-1) \qquad l_2=l(l-1)(l-2) \qquad l_3=l(l-1)(l-2)(l-3).$$

P^{β}_{N} means Jacobi polynomials $P^{(\gamma,\beta)}_{N}(x)$ with $x=\cos 2\psi$ and $\gamma=|\nu-\kappa|$ standardised as usual (*Higher transcendental functions* 1953). As an example of the notation used consider the function $G^{\alpha=1}_{\varkappa=1}$ for $L=2$ in table 1. Its explicit form is as follows:

$$G^{1}_{1}(\psi)=(-1)^n(\sin\psi)^{|\nu-1|}l[\tfrac{1}{2}P^{(|\nu-1|,1)}_{n-1}(\cos 2\psi)-(l+1)P^{(|\nu-1|,2)}_{n-1}(\cos 2\psi)]$$

with

$$n=\tfrac{1}{4}(K+2)-\tfrac{1}{2}|\nu-1| \qquad l=\tfrac{1}{4}(K+2)+\tfrac{1}{2}|\nu-1|.$$

5. Conclusion

In conclusion we write down the general formula for $f^{\alpha\nu}_{KLM}$. In the case of even $K-L$

$$f^{\alpha\nu}_{KLM}(\psi)=[(L+M)!(L-M)!]^{1/2}[(2L)!]^{-1/2}\sum_{\varkappa=-L/2}^{L/2}\varphi^{\varkappa}_{LM}(\psi)\times G^{\varkappa\alpha\nu}_{KL}(\psi) \qquad (19)$$

Table 1. Even $K-L$.

$\underline{L=0}$

$$f^{\alpha=0,\nu}_{K00}=(-1)^n(\sin\psi)^{|\nu|}P^{(|\nu|,0)}_n \qquad n=\tfrac{1}{4}K-\tfrac{1}{2}|\nu|$$

$\underline{L=1}$

$$f^{\alpha\nu}_{K1,\pm1}=\pm[\cos(\psi/2)G^{\alpha}_{\pm1/2}+\sin(\psi/2)G^{\alpha}_{\mp1/2}]$$

$$\alpha=\tfrac{1}{2} \qquad g_{1/2}=lP^1_n \qquad g_{-1/2}=lP^1_{n-1}$$

$\underline{L=2}$

$$f^{\alpha\nu}_{K2,\pm2}=s_1G^{\alpha}_{\pm1}+s_2G^{\alpha}_0+s_3G^{\alpha}_{\mp1}$$

$$f^{\alpha\nu}_{K20}=-1/\sqrt{6}[2s_2(G^{\alpha}_{-1}+G^{\alpha}_1)+G^{\alpha}_0]$$

$$\alpha=1 \qquad g_1=l[\tfrac{1}{2}P^1_{n-1}-(l+1)P^2_{n-1}]$$

$$g_{-1}=l_1P^2_{n-2} \qquad g_0=2l_1P^2_{n-1}$$

$$\alpha=0 \qquad g_{\pm1}=l_1P^2_{n-1} \qquad g_0=l[(l-1)P^2_{n-2}+\tfrac{1}{2}P^1_{n-1}-(l+1)P^2_{n-1}]$$

$\underline{L=3}$

$$f^{\alpha\nu}_{K3,\pm3}=\pm(s_1G^{\alpha}_{\pm3/2}+s_2G^{\alpha}_{\pm1/2}+s_3G^{\alpha}_{\mp1/2}+s_4G^{\alpha}_{\mp3/2})$$

$$f^{\alpha\nu}_{K3,\pm1}=\mp1/\sqrt{15}[3s_2G^{\alpha}_{\pm3/2}+(s_1+2s_3)G^{\alpha}_{\pm1/2}+(s_4+2s_2)G^{\alpha}_{\mp1/2}+3s_3G^{\alpha}_{\mp3/2}]$$

$$\alpha=\tfrac{3}{2} \qquad g_{3/2}=l_1[\tfrac{3}{2}P^2_{n-1}-(l+1)P^3_{n-1}] \qquad g_{-3/2}=l_2P^3_{n-3}$$

$$g_{1/2}=3l_1[\tfrac{1}{2}P^2_{n-2}-(l+1)P^3_{n-2}] \qquad g_{-1/2}=3l_2P^3_{n-2}$$

$$\alpha=\tfrac{1}{2} \qquad g_{3/2}=l_1[\tfrac{1}{2}P^2_{n-2}-(l+1)P^3_{n-2}] \qquad g_{-3/2}=l_2P^3_{n-2}$$

$$g_{1/2}=l_1[2(l-2)P^3_{n-2}+\tfrac{3}{2}P^2_{n-1}-(l+1)P^3_{n-1}]$$

$$g_{-1/2}=l_1[(l-2)P^3_{n-3}+P^2_{n-2}-2(l+1)P^3_{n-2}]$$

$\underline{L=4}$

$$f^{\alpha\nu}_{K4,\pm4}=s_1G^{\alpha}_{\pm2}+s_2G^{\alpha}_{\pm1}+s_3G^{\alpha}_0+s_4G^{\alpha}_{\mp1}+s_5G^{\alpha}_{\mp2}$$

$$f^{\alpha\nu}_{K4,\pm2}=-1/\sqrt{28}[4s_2G^{\alpha}_{\pm2}+(s_1+3s_3)G^{\alpha}_{\pm1}+2(s_2+s_4)G^{\alpha}_0+(s_5+3s_3)G^{\alpha}_{\mp1}+4s_4G^{\alpha}_{\mp2}]$$

$$f^{\alpha\nu}_{K40}=1/\sqrt{70}[6s_3(G^{\alpha}_2+G^{\alpha}_{-2})+3(s_2+s_4)(G^{\alpha}_1+G^{\alpha}_{-1})+(1+2s_3)G^{\alpha}_0]$$

$$\alpha=2 \qquad g_2=l_1[\tfrac{3}{4}P^2_{n-2}-3(l+1)P^3_{n-2}+(l+1)(l+2)P^4_{n-2}]$$

$$g_{-2}=l_3P^4_{n-4} \qquad g_0=6l_2[\tfrac{1}{2}P^3_{n-3}-(l+1)P^4_{n-3}]$$

$$g_1=4l_2[\tfrac{3}{2}P^3_{n-2}-(l+1)P^4_{n-2}] \qquad g_{-1}=4l_3P^4_{n-3}$$

$$\alpha=1 \qquad g_2=l_2[\tfrac{3}{2}P^3_{n-2}-(l+1)P^4_{n-2}] \qquad g_{-2}=l_3P^4_{n-3}$$

$$g_1=3l_2[\tfrac{1}{2}P^3_{n-3}-(l+1)P^4_{n-3}]+l_1[\tfrac{3}{4}P^2_{n-2}-3(l+1)P^3_{n-2}+(l+1)(l+2)P^4_{n-2}]$$

$$g_{-1}=l_2[(l-3)P^4_{n-4}+\tfrac{3}{2}P^3_{n-3}-3(l+1)P^4_{n-3}]$$

$$g_0=3l_3P^4_{n-3}+3l_2[\tfrac{3}{2}P^3_{n-2}-(l+1)P^4_{n-2}]$$

$$\alpha=0 \qquad g_{\pm2}=l_2[\tfrac{1}{2}P^3_{n-3}-(l+1)P^4_{n-3}]$$

$$g_{\pm1}=2l_2[(l-3)P^4_{n-3}+\tfrac{3}{2}P^3_{n-2}-(l+1)P^4_{n-2}]$$

$$g_0=l_3P^4_{n-4}+4l_2[\tfrac{1}{2}P^3_{n-3}-(l+1)P^4_{n-3}]$$

$$+l_1[\tfrac{3}{4}P^2_{n-2}-3(l+1)P^3_{n-2}+(l+1)(l+2)P^4_{n-2}]$$

Table 2. Odd $K-L$.

$\underline{L=1}$		
$f_{K10}^{\alpha=0,\nu}=(-1)^n(K/2+\lvert\nu\rvert+1)/2\cos\psi(\sin\psi)^{\lvert\nu\rvert}P_{n-1}^{(\lvert\nu\rvert,1)}(\cos 2\psi)$		$n=\frac{1}{4}(K+2)-\frac{1}{2}\lvert\nu\rvert$
$\underline{L=2}$		
$f_{K2,\pm 1}^{\alpha\nu}=\pm 1/\sqrt{2}\cos\psi[\cos(\psi/2)G_{\pm 1/2}^{\alpha}+\sin(\psi/2)G_{\mp 1/2}^{\alpha}]$		
$\alpha=\frac{1}{2}$	$g_{1/2}=l_1P_{n-1}^2$	$g_{-1/2}=l_1P_{n-2}^2$
$\underline{L=3}$		
$f_{K3,\pm 2}^{\alpha\nu}=1/\sqrt{3}\cos\psi(s_1G_{\pm 1}^{\alpha}+s_2G_0^{\alpha}+s_3G_{\mp 1}^{\alpha})$		
$f_{K30}^{\alpha\nu}=-1/\sqrt{10}\cos\psi[2s_2(G_1^{\alpha}+G_{-1}^{\alpha})+G_0^{\alpha}]$		
$\alpha=1$	$g_1=l_1[\frac{1}{2}P_{n-2}^2-(l+1)P_{n-2}^3]$	
	$g_{-1}=l_2P_{n-3}^3$	$g_0=2l_2P_{n-2}^3$
$\alpha=0$	$g_{\pm 1}=l_2P_{n-2}^3$	$g_0=l_1[(l-2)P_{n-3}^3+\frac{1}{2}P_{n-2}^2-(l+1)P_{n-2}^3]$
$\underline{L=4}$		
$f_{K4,\pm 3}^{\alpha\nu}=\pm\frac{1}{2}\cos\psi[s_1G_{\pm 3/2}^{\alpha}+s_2G_{\pm 1/2}^{\alpha}+s_3G_{\mp 1/2}^{\alpha}+s_4G_{\mp 3/2}^{\alpha}]$		
$f_{K4,\pm 1}^{\alpha\nu}=\mp 1/\sqrt{28}\cos\psi[3s_2G_{\pm 3/2}^{\alpha}+(s_1+2s_3)G_{\pm 1/2}^{\alpha}+(s_4+2s_2)G_{\mp 1/2}^{\alpha}+3s_3G_{\mp 3/2}^{\alpha}]$		
$\alpha=\frac{3}{2}$	$g_{3/2}=l_2[\frac{3}{2}P_{n-2}^3-(l+1)P_{n-2}^4]$	$g_{-3/2}=l_3P_{n-4}^4$
	$g_{1/2}=3l_2[\frac{1}{2}P_{n-3}^3-(l+1)P_{n-3}^4]$	$g_{-1/2}=3l_3P_{n-3}^4$
$\alpha=\frac{1}{2}$	$g_{3/2}=l_2[\frac{1}{2}P_{n-3}^3-(l+1)P_{n-3}^4]$	$g_{-3/2}=l_3P_{n-3}^4$
	$g_{1/2}=l_2[2(l-3)P_{n-3}^4+\frac{3}{2}P_{n-2}^3-(l+1)P_{n-2}^4]$	
	$g_{-1/2}=l_2[(l-3)P_{n-4}^4+P_{n-3}^3-2(l+1)P_{n-3}^4]$	

$$\varphi_{LM}^{\varkappa}(\psi)=t(L,\varkappa)\sum_{k_1k_2n_1n_2}[\cos(\psi/2)]^{k_1+n_2}[\sin(\psi/2)]^{k_2+n_1}(k_1!k_2!n_1!n_2!)^{-1} \tag{20}$$

$$t(L,\varkappa)=(-1)^{\frac{1}{2}(L-M)}(L/2+\varkappa)!(L/2-\varkappa)!$$

$$G_{KL}^{\varkappa\alpha\nu}(\psi)=(-1)^n(\sin\psi)^{\lvert\nu-\varkappa\rvert}\sum_{m=m_1}^{m_2}a(m)\sum_{i=0}^{m}b(i,m)P_{n-L+m+\varepsilon}^{(\lvert\nu-\varkappa\rvert,L-m+i)}(\cos 2\psi) \tag{21}$$

$$m_1=\tfrac{1}{2}(\lvert\alpha+\varkappa\rvert-\varepsilon)\qquad m_2=\tfrac{1}{2}(L-\lvert\alpha-\varkappa\rvert-\varepsilon) \tag{22}$$

$$a(m)=\binom{\frac{1}{2}L+\alpha}{m+\frac{1}{2}(\alpha+\varkappa+\varepsilon)}\binom{\frac{1}{2}L-\alpha}{m-\frac{1}{2}(\alpha+\varkappa-\varepsilon)}\frac{m!\Gamma(m+\varepsilon+\frac{1}{2})}{(l+m-L)!}$$
$$b(i,m)=(-1)^i(l+i)![(m-i)!i!\Gamma(i+\varepsilon+\tfrac{1}{2})]^{-1}. \tag{23}$$

In the above expression $\varepsilon=0$ if the $\alpha+\varkappa$ value is even and $\varepsilon=1$ if the $\alpha+\varkappa$ value is odd†. The n and l quantities are defined in (18a). The summation in (20) is under the conditions

$$k_1+n_1=\tfrac{1}{2}L+\varkappa\qquad k_2+n_2=\tfrac{1}{2}L-\varkappa$$
$$k_1+k_2=\tfrac{1}{2}(L+M)\qquad n_1+n_2=\tfrac{1}{2}(L-M). \tag{24}$$

Three out of the four conditions of (24) are independent, so in the sum of (20) we have only one independent variable.

†In the paper by Mukhtarova and Efros (1983) there is a misprint in the definition of ε.

The expression for $f^{\alpha\nu}_{KLM}$ in the case of odd $K-L$ is obtained from (19)-(24) by means of the following substitutions. In (19), in the summation, the limits become $L \to L-1$. For the $\varphi^{\varkappa}_{LM}(\psi)$ function defined by (20) and (24) the substitution should be $\varphi^{\varkappa}_{LM}(\psi) \to \sqrt{2}\cos\psi\varphi^{\varkappa}_{L-1,M}(\psi)$. In addition, in the expression for m_2 in (22) and in the binomial coefficients of (23) substitution $L \to L-1$ should be made. The n and l values in (21) and (23) are defined in (18b).

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